Stability of Similarity Solutions by Two-Equation Models of Turbulence

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The stability of the similarity solution of a turbulent round and plane jet with zero outer velocity is investigated. Two solutions of the equations with different modeling constants are analyzed. One is Jones' and Launder's k-ϵ model and the other is Rotta's k-ℓ model. The ℓ equation is transformed into the form of an ϵ equation in order to permit the comparison of the different constants of the models. In addition, an equation for the turbulent viscosity is derived. The mathematical properties of the equations are discussed. It is shown that, in the limit of vanishing laminar viscosity, the parabolic equations degenerate to a form which exhibits discontinuities and has a wavelike solution. This property is discussed by comparison with Zeldovich's thermal waves. The similarity equations are derived. The stiffness of the equations is shown by an asymptotic expansion around the limit "production equals dissipation." The similarity solution is obtained by a shooting technique. A linear stability analysis of the similarity solution is performed. The linear perturbation of this solution yields by a discretisation technique at least one eigenvalue with a positive real part, showing unstable solutions.

Introduction

IT is widely known that numerical integration techniques using two-equation models of turbulence exhibit severe stability problems. It is also observed that the reduction in step size does not avoid the stability problems, but rather increases them. It seems that a certain amount of numerical damping is necessary to obtain stable solutions. In free shear flows of the boundary-layer type, the problem is sometimes circumvented by solving the governing equations in transformed coordinate systems where the normalized stream function varying between 0 and 1 appears as independent normal coordinate.[1] This technique is not applicable to elliptic flows with arbitrary boundary conditions, where the equation must be solved in the physical plane. When the boundary conditions have to be applied at large distance from the region of main turbulence production (for instance, with a jet flow into a chamber), stable numerical solutions are very difficult to obtain.

There appears to be a disproportion between the large number of numerical applications of two-equation models of turbulence and of more fundamental investigations into their mathematical structure. These equations are, of course, highly nonlinear and are generally solved in at least two space dimensions. In this paper, we will point out certain properties of the equations that will lead to disasters in numerical computations if they are overlooked. It is the purpose of the paper to show that the stability difficulties are not likely to originate from numerical schemes, but are a property of the equations.

Governing Equations

Using the boundary-layer approximation, the governing equations for the mean velocities and the turbulent kinetic energy are written as

Continuity:

$$\frac{\partial u}{\partial x}+\frac{1}{y^j}\frac{\partial (y^j v)}{\partial y}=0 \tag{1}$$

Momentum:

$$uy^j\frac{\partial u}{\partial x}+vy^j\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}\left[(\nu_t+\nu)y^j\frac{\partial u}{\partial y}\right] \tag{2}$$

Turbulent kinetic energy:

$$uy^j\frac{\partial k}{\partial x}+vy^j\frac{\partial k}{\partial y}=\frac{\partial}{\partial y}\left[\frac{(\nu_t+\nu)}{Pr_k}y^j\frac{\partial k}{\partial y}\right] +y^j\nu_t\left(\frac{\partial u}{\partial y}\right)^2-y^j\epsilon \tag{3}$$

Plane jet: $j=0$, Round jet: $j=1$

The kinetic energy equation was first derived by Prandtl.[2] It contains as unknown quantities the turbulent viscosity ν_t and the turbulent dissipation ϵ. Prandtl has shown that both can be related to at least one additional quantity, a turbulence length scale ℓ by

$$\nu_t=c_1\sqrt{k}\ell,\quad =c(k^{3/2}/\ell) \tag{4}$$

Later, Rotta[3] developed a balance equation for ℓ on the basis of an equation for a two point correlation function. His more recent formulation has the form[4]

$$\begin{aligned}uy^j\frac{\partial k\ell}{\partial x}+vy^j\frac{\partial k\ell}{\partial y}&=\frac{\partial}{\partial y}\left[\frac{(\nu_t+\nu)}{Pr_\ell}y^j\left(\ell\frac{\partial k}{\partial y}+\alpha_L k\frac{\partial \ell}{\partial y}\right)\right.\\&+y^j\left\{\nu_t\frac{\partial u}{\partial y}\left[\zeta\ell\frac{\partial u}{\partial y}+\zeta_3\ell^3\left(\frac{2j}{y}\frac{\partial^2 u}{\partial y^2}+\frac{\partial^3 u}{\partial y^3}\right)\right]\right.\\&\left.+\frac{\partial}{\partial y}\left(\nu_t\frac{\partial u}{\partial y}\right)\zeta_2\ell^3\left(\frac{j}{y}\frac{\partial u}{\partial y}+\frac{\partial^2 u}{\partial y^2}\right)\right\}\\&-y^jc_\ell ck^{3/2}-y^jc_2k\ell\frac{\partial u}{\partial x}\end{aligned} \tag{5}$$

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where

$$c = c_1^3 = 0.165, \quad c_2 = 1, \quad c_\ell = 0.8$$

$$Pr_k = Pr_\ell = c_1/k_q = 0.6856, \quad k_q = 0.8, \quad \alpha_\ell = 0.387$$

$$\zeta = 0.98, \quad \zeta_2 = 1.2, \quad \zeta_3 = -1.5$$

Alternatives to a length scale equation are equations for ϵ or ν_t. An extensively tested and widely used model is the k-ϵ model by Jones and Launder.[5] For sufficiently large turbulent Reynolds numbers, the ϵ equation is written as

$$uy^j \frac{\partial \epsilon}{\partial x} + vy^j \frac{\partial \epsilon}{\partial y} = \frac{\partial}{\partial y}\left[\frac{(\nu_t+\nu)}{Pr_\epsilon} y^j \frac{\partial \epsilon}{\partial y}\right] + y^j c_{\epsilon 1}\nu_t \frac{\epsilon}{k}\left(\frac{\partial u}{\partial y}\right)^2 - y^j c_{\epsilon 2}\epsilon^2/k \tag{6}$$

where

$$\nu_t = c_D(k^2/\epsilon), \quad c_D = c_1 c = 0.09$$

$$c_{\epsilon 1} = 1.44, \quad c_{\epsilon 2} = 1.92$$

$$Pr_k = 1.0, \quad Pr_\epsilon = 1.3$$

Using the algebraic relation between ℓ, ϵ, and k, Rotta's ℓ equation can be cast into the same form as the ϵ equation, where the following additional terms appear on the right-hand side:

(Additional terms) =

$$\begin{aligned} &-y^j c_{\epsilon 3}\nu_t \frac{k^2}{\epsilon}\frac{\partial u}{\partial y}\left(\frac{\partial^3 u}{\partial y^3} + \frac{2j}{y}\frac{\partial^2 u}{\partial y^2}\right) \\ &-y^j \zeta_{22}\nu_t \frac{\partial}{\partial y}\left(\nu_t \frac{\partial u}{\partial y}\right)\left(\frac{j}{y}\frac{\partial u}{\partial y} + \frac{\partial^2 u}{\partial y^2}\right) \\ &-c_{\epsilon 4}\frac{\epsilon}{k}\frac{\partial}{\partial y}\left[(\nu_t+\nu)y^j \frac{\partial k}{\partial y}\right] - y^j c_{\epsilon 5}\nu_t \frac{\epsilon}{k^2}\left(\frac{\partial k}{\partial y}\right)^2 \\ &+y^j c_{\epsilon 6}\nu_t \frac{1}{k}\frac{\partial k}{\partial y}\frac{\partial \epsilon}{\partial y} - y^j c_{\epsilon 7}\nu_t \frac{1}{\epsilon}\left(\frac{\partial \epsilon}{\partial y}\right)^2 \\ &+c_2 \epsilon \frac{\partial u}{\partial x} \end{aligned} \tag{7}$$

where

$$Pr_k = Pr_\ell = 0.6856, \quad Pr_\epsilon^* = Pr_\ell/\alpha_\ell = 1.77$$

$$c_{\epsilon 1} = 5/2 - \zeta = 1.52, \quad c_{\epsilon 2} = 5/2 - c_\ell = 1.70$$

$$c_{\epsilon 3}^* = \zeta_3 c^2 = -0.041, \quad \zeta_{22} = c^{2/3}\zeta_2 = 0.361$$

$$c_{\epsilon 4}^* = \frac{(1+3\alpha_\ell/2)}{Pr_\ell} - \frac{5}{2Pr_k} = -1.34$$

$$c_{\epsilon 5}^* = \frac{3(1+3\alpha_\ell/2)}{2Pr_\ell} = 3.4579$$

$$c_{\epsilon 6}^* = \frac{(1+3\alpha_\ell/2)}{Pr_\ell} + \frac{5}{2Pr_\epsilon^*} = 3.716$$

$$c_{\epsilon 7}^* = 2/Pr_\epsilon^* = 1.13$$

The occurrence of higher-order velocity derivatives in this equation leads to additional numerical difficulties. It is consistent with the assumption of a single length scale ℓ and a single time scale k/ϵ to replace the velocity derivatives by using the assumption production equals dissipation in the turbulence energy equation,

$$\epsilon = \nu_t\left(\frac{\partial u}{\partial y}\right)^2 \tag{8}$$

leading to

$$\frac{\partial u}{\partial y} = \frac{1}{c^{1/3}}\frac{\epsilon}{k} \tag{9}$$

Then the first two terms in Eq. (7) are to be replaced by

$$-j\nu_t\left[c_{\epsilon 3}\left(\frac{\partial \epsilon}{\partial y} - \frac{\epsilon}{k}\frac{\partial k}{\partial y}\right) + \zeta_{22}\frac{\epsilon}{k}\frac{\partial k}{\partial y}\right]$$

$$Pr_\epsilon = Pr_\epsilon^*/(1 - Pr_\epsilon^* c_{\epsilon 3}) = 0.985$$

$$c_{\epsilon 3} = c^{2/3}\zeta_3 = -0.451$$

$$c_{\epsilon 4} = c_{\epsilon 4}^* - c_{\epsilon 3} = -0.890$$

$$c_{\epsilon 5} = c_{\epsilon 5}^* + 4c_{\epsilon 3} - \zeta_{22} = 1.292$$

$$c_{\epsilon 6} = c_{\epsilon 6}^* + 5c_{\epsilon 3} - \zeta_{22} = 1.099$$

$$c_{\epsilon 7} = c_{\epsilon 7}^* + c_{\epsilon 3} = 0.678$$

It is seen that the modified Rotta model and the k-ϵ model differ only slightly in the common constants Pr_ϵ, $c_{\epsilon 1}$, and $c_{\epsilon 2}$, but that the Rotta model contains additional terms. Voges[6] has compared the Rotta model with other two-equation models of turbulence, including the k-ϵ model, in their capability to "postdict" the boundary-layer data of the 1968 Stanford Conference.[7] He shows that it is, in particular, the last term in Eq. (7) that causes considerable deficiencies. If c_2 is set zero, the predictive power of the Rotta model is similar to that of the k-ϵ model. The third choice is an equation for ν_t rather than one for ℓ or ϵ. Defining

$$\lambda = k^2/\epsilon, \quad \nu_t = c_D\lambda \tag{10}$$

and using again the algebraic relation between ν_t, ϵ, and k, one obtains an equation for

$$\begin{aligned} uy^j \frac{\partial \lambda}{\partial x} + vy^j \frac{\partial \lambda}{\partial y} = {}& \frac{\partial}{\partial y}\left(\frac{\nu_t+\nu}{Pr_\lambda} y^j \frac{\partial \lambda}{\partial y}\right) \\ &+ y^j(2 - c_{\epsilon 1})\nu_t \frac{\lambda}{k}\left(\frac{\partial u}{\partial y}\right)^2 - y^j(2 - c_{\epsilon 2})k \\ &+ c_{\lambda 4}\frac{\lambda}{k}\frac{\partial}{\partial y}\left[(\nu_t+\nu)y^j \frac{\partial k}{\partial y}\right] + y^j c_{\lambda 5}\nu_t \frac{\lambda}{k^2}\left(\frac{\partial k}{\partial y}\right)^2 \\ &+ y^j c_{\lambda 6}\nu_t \frac{1}{k}\frac{\partial k}{\partial y}\frac{\partial \lambda}{\partial y} + y^j c_{\lambda 7}\nu_t \frac{1}{\lambda}\left(\frac{\partial \lambda}{\partial y}\right)^2 \\ &+ jc_{\epsilon 3}\nu_t \frac{\lambda}{k}\left(\frac{\partial k}{\partial y} - \frac{k}{\lambda}\frac{\partial \lambda}{\partial y}\right) + j\nu_t\zeta_{22}\frac{\lambda}{k}\frac{\partial k}{\partial y} \end{aligned} \tag{11}$$

where $c_2 = 0$ has been introduced and the constants are

$$c_{\lambda 4} = 2/Pr_k - 2/Pr_\epsilon + c_{\epsilon 4} = -0.004$$

$$c_{\lambda 5} = c_{\epsilon 5} + 4c_{\epsilon 7} - 2c_{\epsilon 6} - 2/Pr_\epsilon = -0.227$$

$$c_{\lambda 6} = c_{\epsilon 6} - 4c_{\epsilon 7} + 4/Pr_\epsilon = 2.45$$

$$c_{\lambda 7} = c_{\epsilon 7} - 2/Pr_\epsilon = -1.35, \quad Pr_\lambda = Pr_\epsilon = 0.985$$

(The numerical values are for the Rotta model.)

Finally, a general relation between the constants can be obtained by requiring that the universal law at the wall should be satisfied. From

$$u = u_\tau \left(\frac{1}{\kappa} \ln \frac{y u_\tau}{\nu} + c \right) \tag{12}$$

where $\kappa = 0.4$ is von Kármán's constant and Eq. (8) follows

$$k = u_\tau^2 / c^{2/3}, \quad \lambda = \kappa u_\tau y / c_D$$

and from Eq. (11) one obtains

$$c_{\epsilon 2} - c_{\epsilon 1} + \kappa^2 c^{-2/3} (c_{\epsilon 7} - 1/Pr_\epsilon) = 0 \tag{13}$$

This equation will be used to determine $c_{\epsilon 7}$ in the modified Rotta model.

Wave-Like Solutions

A certain class of nonlinear diffusion equations are known to degenerate, if the diffusion coefficient depends nonlinearly on the solution and vanishes at a boundary at infinity. A well-known example of an equation from this class is the so called "porous media equation,"

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v^m}{\partial x^2}, \quad m > 1 \tag{14}$$

or the "thermal wave equation" of Zeldovich[8]

$$\frac{\partial T}{\partial t} = a \frac{\partial}{\partial x} \left(T^n \frac{\partial T}{\partial x} \right), \quad n > 0 \tag{15}$$

with the boundary condition v, $T \to 0$ as $x \to \infty$. Equations (14) and (15) have solutions with singularities in the form of discontinuous derivatives. The mathematical theory for such degenerate parabolic equations has been developed in Refs. 9-13. For instance, the solution of Eq. (15), with a constant temperature T_0 on the boundary $x = 0$, has the form as shown in Fig. 1 (cf. Ref. 8). It is a thermal wave front with a front coordinate x_f proportional to $\sqrt{(aT_0^n t)}$ and a propagation velocity proportional to $\sqrt{(aT_0^n / t}$. Ahead of the front, the temperature remains undisturbed. This is in contrast to the solution of the classical linear heat conduction equation, where the propagation velocity is infinite.

The λ equation turns into a form very similar to eq. (15) for $\nu \to 0$ if the boundary $\lambda = 0$ is applied at $y \to \infty$. It is easily verified that in this case the undisturbed solution $\lambda = 0$ satisfies the equation. Therefore, one may expect a discontinuous derivative to develop at a finite distance y_f. The existence of a singularity in free shear flows for $\nu \to 0$ is, in fact, well known for Prandtl's mixing length model[14] and Rotta's k-ℓ model.[4] For finite but small values of ν, this singularity is smeared out. However, if the boundary conditions are applied at large values of y and ν is small, the degenerate character of the equation may lead to severe stability problems. In many numerical calculations, it seems that the generated numerical viscosity rescues the situation.

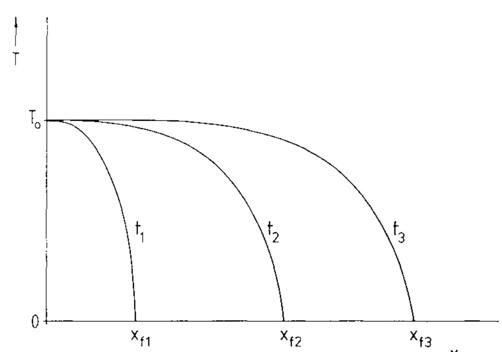

Fig. 1 Schematic solution of Zeldovich's thermal wave.

The Similarity Solution

We introduce the similarity coordinates

$$\xi = (x + x_0)/d, \quad \eta = \gamma y / (\xi d) \tag{16}$$

and the stream function

$$\psi = K_p \xi^n F(\xi, \eta) \tag{17}$$

Plane jet: $n = 1/2$, $K_p = \gamma \nu_{t\mathrm{ref}}$

Round jet: $n = 1$, $K_p = d \nu_{t\mathrm{ref}}$

where $\nu_{t\mathrm{ref}}$ is related to the spreading parameter γ, the nozzle diameter d, the nozzle exit velocity u_0, and the nondimensional centerline decay of the velocity

$$a = \frac{d(u_c/u_0)}{d(1/\xi)^n}$$

as

$$\nu_{t\mathrm{ref}} = u_0 d a / \gamma^2 \tag{18}$$

The continuity equation is satisfied by

$$u = \frac{1}{y^j} \frac{\partial \psi}{\partial y}, \quad v = -\frac{1}{y^j} \frac{\partial \psi}{\partial x} \tag{19}$$

Introducing these definitions into the momentum equation, one obtains for the nondimensional velocity $U = u/u_c = F_\eta / \eta^j$,

$$\xi \left(U \frac{\partial F_\eta}{\partial \xi} - \frac{\partial F}{\partial \xi} \frac{\partial U}{\partial \xi} \right) - n \frac{\partial}{\partial \eta} (FU) = \frac{\partial}{\partial \eta} \left(C \eta^j \frac{\partial U}{\partial \eta} \right) \tag{20}$$

Here, the Chapman-Rubesin parameter

$$C = (\nu_t + \nu) / \nu_{t\mathrm{ref}} = \Lambda + c_\nu \tag{21}$$

contains contributions from the turbulent as well as the laminar viscosity. Since $u_0 d / \nu_{t\mathrm{ref}}$ is typically 70 in a round jet, c_ν is of the order $70/Re$.

In the similarity region $(\partial / \partial \xi = 0)$, the momentum equation can be integrated once using the boundary conditions $F = 0$, $U = 1$ as $\eta = 0$ and $U \to 0$ as $\eta \to \infty$ to obtain

$$-FUn = C \eta^j \frac{\partial U}{\partial \eta} \tag{22}$$

Introducing the nondimensional quantities,

$$\kappa = \frac{k \xi^{2n}}{b u_0^2}, \quad \Lambda = \frac{\lambda \xi^{n-1}}{u_0 d c} = \frac{c_D \lambda \xi^{n-1}}{\nu_{t\mathrm{ref}}} \tag{23}$$

one obtains the equation for the turbulent kinetic energy and turbulent viscosity,

$$F_\eta \left(\xi \frac{\partial \kappa}{\partial \xi} - 2n\kappa \right) - \left(nF + \xi \frac{\partial F}{\partial \xi} \right) \frac{\partial \kappa}{\partial \eta}$$
$$= \frac{\partial}{\partial \eta} \left(\frac{C \eta^j}{Pr_k} \frac{\partial \kappa}{\partial \eta} \right) + \eta^j E_1 \Lambda \left(\frac{\partial U}{\partial \eta} \right)^2 - \eta^j E_2 \frac{\kappa^2}{\Lambda} \tag{24}$$

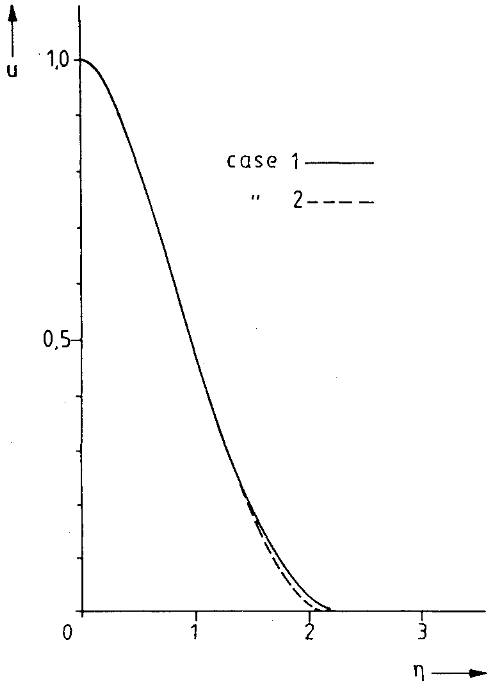

Fig. 2 Similar solution of the velocity for the plane jet.

with $E_1 = a^2/b$ and $E_2 = b/(ac)$,

$$F_\eta\left(\xi\frac{\partial\Lambda}{\partial\xi}+(1-n)\Lambda\right)-\left(nF+\xi\frac{\partial F}{\partial\xi}\right)\frac{\partial\Lambda}{\partial\eta}$$

$$=\frac{\partial}{\partial\eta}\left(\frac{C\eta^j}{Pr_\lambda}\frac{\partial\Lambda}{\partial\eta}\right)+\eta^jE_1(2-c_{\epsilon1})\frac{\Lambda^2}{\kappa}\left(\frac{\partial U}{\partial\eta}\right)^2$$

$$-\eta^jE_2(2-c_{\epsilon2})\kappa+c_{\lambda4}\frac{\Lambda}{\kappa}\frac{\partial}{\partial\eta}\left(C\eta^j\frac{\partial\kappa}{\partial\eta}\right)$$

$$+\eta^jc_{\lambda5}\frac{\Lambda^2}{\kappa^2}\left(\frac{\partial\kappa}{\partial\eta}\right)^2+\eta^jc_{\lambda6}\frac{\Lambda}{\kappa}\frac{\partial\kappa}{\partial\eta}\frac{\partial\Lambda}{\partial\eta}$$

$$+\eta^jc_{\lambda7}\left(\frac{\partial\Lambda}{\partial\eta}\right)^2+j\left[c_{\epsilon3}\frac{\Lambda^2}{\kappa}\left(\frac{\partial\kappa}{\partial\eta}-\frac{\kappa}{\Lambda}\frac{\partial\Lambda}{\partial\eta}\right)\right.$$

$$\left.+\zeta_{22}\frac{\Lambda^2}{\kappa}\frac{\partial\kappa}{\partial\eta}\right] \quad (25)$$

With the boundary conditions,

$$\text{At } \eta=0\text{: } \kappa=1,\ \frac{\partial\kappa}{\partial\eta}=0$$

$$\Lambda=1,\ \frac{\partial\Lambda}{\partial\eta}=0$$

$$\text{At } \eta\to\infty\text{: } \kappa\to0,\quad \Lambda\to0 \quad (26)$$

The system of ordinary differential equations obtained in the similarity region has to satisfy two more boundary conditions than necessary. Therefore, it represents an eigenvalue problem for two parameters, for instance, E_1 and E_2 related to the centerline decay coefficients b and c. In the numerical method, however, it was found necessary to prescribe E_1 and consider E_2 and $c_{\epsilon2}$ as eigenvalues.

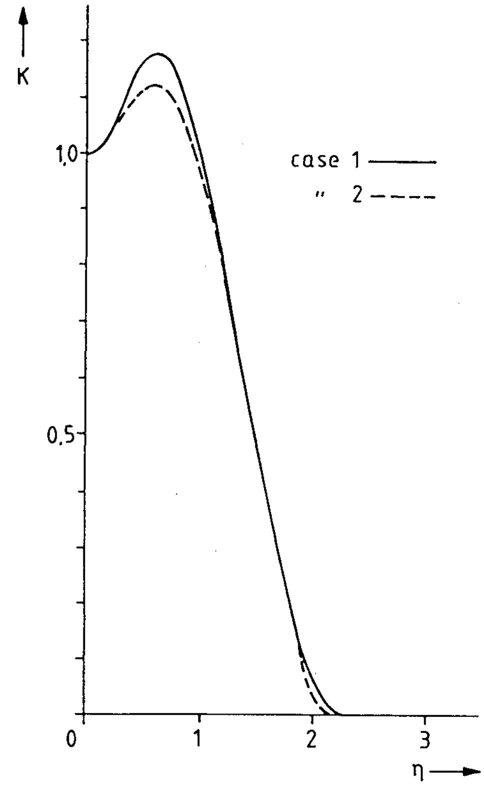

Fig. 3 Similar solution of the kinetic energy of turbulence for the plane jet.

Table 1 Comparison of k-ϵ and k-ℓ models

Parameter	Case 1[a]	Case 2[b]	Parameter	Case 1[a]	Case 2[b]
Pr_ϵ	1.3	0.985	$c_{\lambda5}$	−1.54	−1.93
Pr_k	1.0	0.685	$c_{\lambda6}$	3.08	4.15
c_λ	0.025	0.05	$c_{\lambda7}$	−1.54	−1.778
$c_{\epsilon1}$	1.44	1.52	ζ_{22}	0	0.36
$c_{\epsilon2}$ ($j=0$)	1.937	1.96	E_1 ($j=0$)	14.707	14.707
$c_{\epsilon2}$ ($j=1$)	1.918	1.926	E_1 ($j=1$)	9.566	9.566
$c_{\epsilon3}$	0	−0.45	E_2 ($j=0$)	1.724	1.765
$c_{\lambda4}$	0.46	−0.003	E_2 ($j=1$)	2.177	1.987

[a] Jones and Launder's k-ϵ model. [b] Rotta's k-ℓ model.

The similarity equations were solved by a multiple shooting method using a program package developed by Burlisch and Stoer.[15] The outer boundary condition was imposed at $\eta=6$. In particular, the solution for the k-ϵ model was extremely difficult to obtain. This is seen by considering the limit $\eta\to\infty$ with $\kappa=0$ prescribed. Then, the diffusion term and the term containing the coefficient $c_{\epsilon7}$ balance and one obtains in the limit $c_\nu=0$,

$$\Lambda\simeq(\eta-\eta_f)^{(1/c_{\epsilon7}Pr_\epsilon)} \text{ for the Rotta model}$$

$$\Lambda\simeq\exp\eta \text{ for the } k\text{-}\epsilon \text{ model}$$

For $c_{\epsilon7}>0$, the Rotta model shows the discontinuity at η_f discussed in the preceding paragraph, while the k-ϵ model shows an increase in infinity as $\eta\to\infty$.

The stiffness of the equation can be seen by considering the asymptotic limit of large values of E_1 with $E_2/E_1=0(1)$. This limit is justified for large η by the fact that $\partial U/\partial\eta$ is small and E_1 is of order 10. Then, to leading order, there results from the k equation the approximation of Eq. (8). The same limit, applied to the ϵ equation yields a value of ϵ that is $c_{\epsilon1}/c_{\epsilon2}$ times that obtained from the k equation. This contradiction can be resolved only by requiring that $c_{\epsilon1}/c_{\epsilon2}$

differs from one only by an amount of order $\mathcal{O}(E_1^{-1})$, which is at least not evident. The existence of two competing fixpoints for large E_1 adds certainly to the numerical difficulties posed by the k-ϵ model.

Typical profiles for both models are shown in Figs. 2-4 for the plane jet and in Figs. 5-7 for the round jet. The corresponding sets of parameters are given in Table 1. Using Eqs. (18) and (23) as well as the momentum integral

$$I=\int_0^\infty U^2\eta^j \mathrm{d}\eta=\frac{\gamma^{j+1}}{(j+1)2^{j+1}a^2} \tag{27}$$

the spreading parameter and decay coefficients were obtained from

$$\gamma=2[I(j+1)a^2]^{1/(j+1)}$$

$$a=\left[\frac{E_1E_2}{4c_D[I(j+1)]^{2/j+1}}\right]^{(j+1)/4}$$

$$b=\frac{E_2}{4c_D[I(j+1)a^{(1-j)}]^{2/(j+1)}}$$

A Global Stability Argument

In this section, we want to develop a global stability argument that will be useful for the interpretation of the subsequent stability results. We will only consider the k-ϵ model and write the ϵ equation for the normalized quantity,

$$E=\epsilon c d\xi^{3n+1}/(b^2u_0^3) \tag{28}$$

as

$$F_\eta\left[\xi\frac{\partial E}{\partial \xi}-(3n+1)E\right]-\left(Fn+\xi\frac{\partial F}{\partial \eta}\right)\frac{\partial E}{\partial \eta}$$
$$=\frac{\partial}{\partial \eta}\left(\frac{C\eta^j}{Pr_\epsilon}\frac{\partial E}{\partial \eta}\right)+\eta^j c_{\epsilon 1}E_1\kappa\left(\frac{\partial U}{\partial \eta}\right)^2-\eta^j c_{\epsilon 2}E_2\frac{E^2}{\kappa} \tag{29}$$

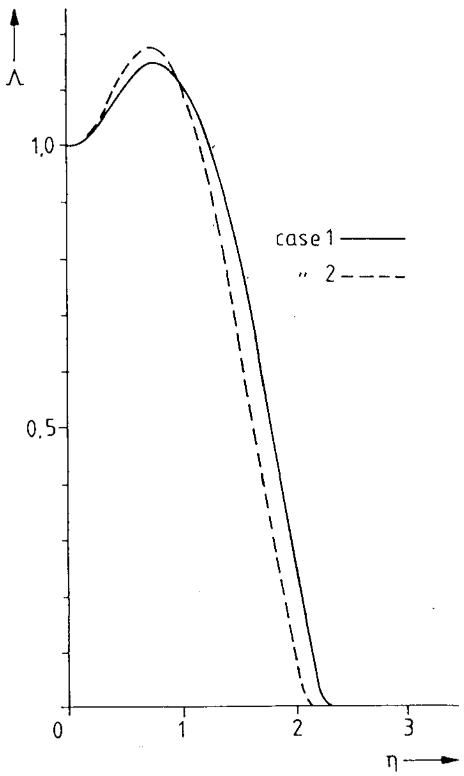

Fig. 4 Similar solution of the turbulent viscosity for the plane jet.

We assume $U=U_s(\eta)$ and $F=F_s(\eta)$ to be independent of ξ and introduce for κ and E small perturbations around the similarity solution,

$$\kappa=\kappa_s(\eta)+\kappa'(\xi,\eta)$$
$$E=E_s(\eta)+E'(\xi,\eta)$$
$$C=C_s(\eta)+C'(\xi,\eta) \tag{30}$$

where $C'=2\kappa'\kappa_s/E_s-E'\kappa_s^2/E_s^2$.

Introducing these into Eqs. (24) and (28) yields the perturbation equations

$$\frac{\partial(\eta^jU_s\kappa')}{\partial \ln\xi}-2n\eta^jU_s\kappa'-nF_s\frac{\partial \kappa'}{\partial \eta}=\frac{\partial}{\partial \eta}\left(\frac{C_s\eta^j}{Pr_k}\frac{\partial \kappa'}{\partial \eta}\right)$$
$$+\frac{\partial}{\partial \eta}\left(\frac{C'\eta^j}{Pr_k}\frac{\partial \kappa_s}{\partial \eta}\right)+\eta^jE_1C'\left(\frac{\partial U_s}{\partial \eta}\right)^2-\eta^jE_2E' \tag{31}$$

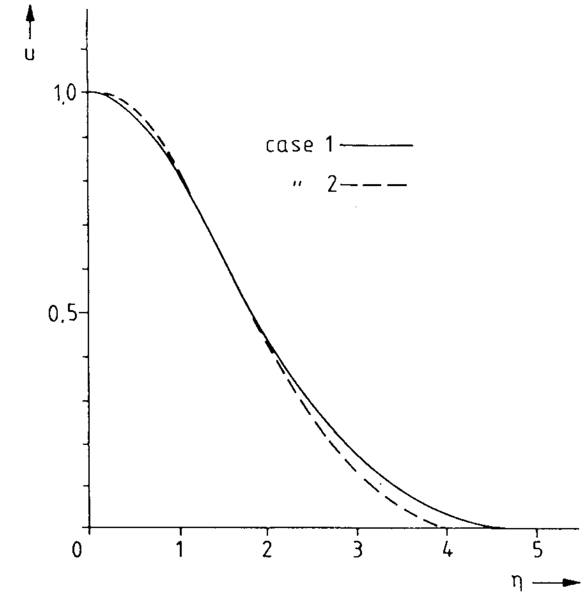

Fig. 5 Similar solution of the velocity for the round jet.

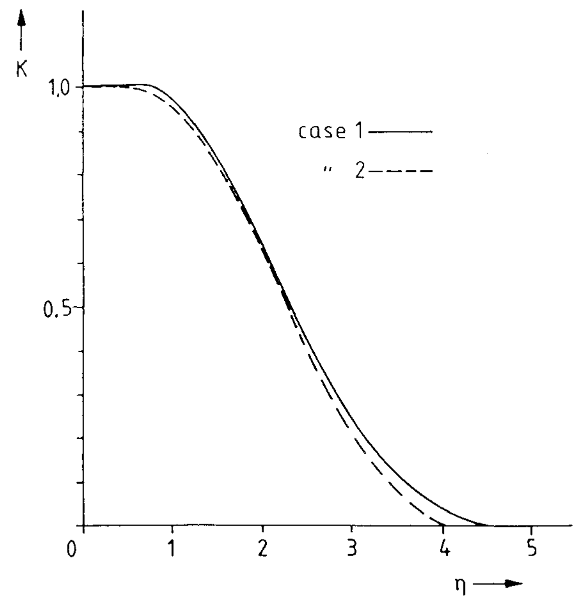

Fig. 6 Similar solution of the kinetic energy of turbulence for the round jet.

and

$$\frac{\partial(\eta^j U_s E')}{\partial \ln\xi} - (3n+1)\eta^j U_s E' - nF_s \frac{\partial E'}{\partial\eta}$$
$$= \frac{\partial}{\partial\eta}\left(\frac{C_s\eta^j}{Pr_\epsilon}\frac{\partial E'}{\partial\eta}\right) + \frac{\partial}{\partial\eta}\left(\frac{C'\eta^j}{Pr_\epsilon}\frac{\partial E_s}{\partial\eta}\right)$$
$$+\eta^j c_{\epsilon1}E_1\kappa'\left(\frac{\partial U_s}{\partial\eta}\right)^2 - \eta^j c_{\epsilon2}E_2\left(2\frac{E'E_s}{\kappa_s} - \frac{\kappa' E_s}{\kappa_s^2}\right) \quad (32)$$

In order to simplify the analysis, we introduce the approximation (justified by the similarity solution) that κ_s may be set equal to E_s and that

$$\left(\frac{\partial U_s}{\partial\eta}\right)^2 = \alpha\frac{E_2}{E_1} \quad (33)$$

where α is a constant of order one to be estimated from the similarity solution. Integrating over the equations and introducing the definitions

$$I_1 = \int_0^\infty \eta^j U_s\kappa' d\eta, \quad I_2 = \int_0^\infty \eta^j\kappa' d\eta$$
$$I_3 = \int_0^\infty \eta^j U_s E' d\eta, \quad I_4 = \int_0^\infty \eta^j E' d\eta \quad (34)$$

the diffusion terms and part of the convective terms drop out and one obtains, with $\kappa'(0)=\kappa'(\infty)=E'(0)=E'(\infty)=0$, the equations

$$\frac{\partial I_1}{\partial \ln\xi} - nI_1 = E_2[2\alpha I_2 - (1+\alpha)I_4]$$
$$\frac{\partial I_3}{\partial \ln\xi} - (2n+1)I_3 = E_2(c_{\epsilon1}\alpha + c_{\epsilon2})I_2 - 2c_{\epsilon2}I_4 \quad (35)$$

Since U_s is always positive, it lies within the accuracy of the previous assumptions to relate I_1 with I_2 and I_3 with I_4 by $I_2 = mI_1$, $I_4 = mI_3$, with m slightly larger than one. Then, one obtains a solvable system of two linear ordinary differential equations with the solution

$$I_1 = A_1\xi^\mu, \quad I_3 = A_3\xi^\mu \quad (36)$$

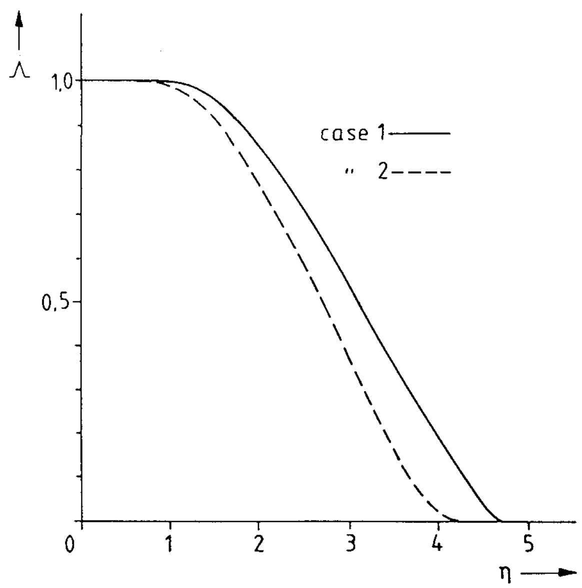

Fig. 7 Similar solution of the turbulent viscosity for the round jet.

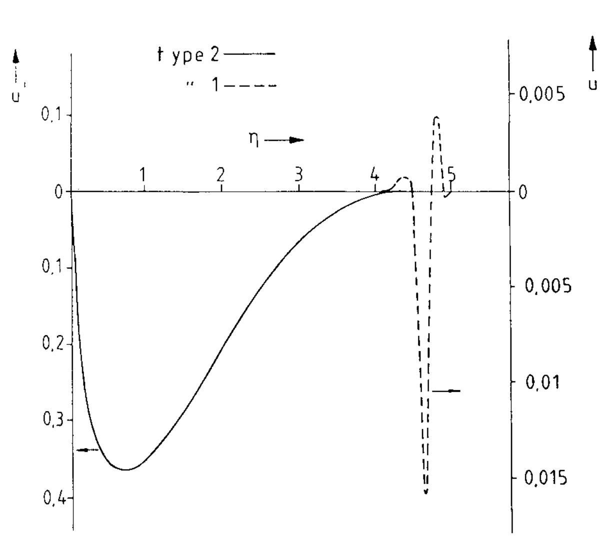

Fig. 8 Unstable eigenvectors of the case 1 velocity profile for the round jet.

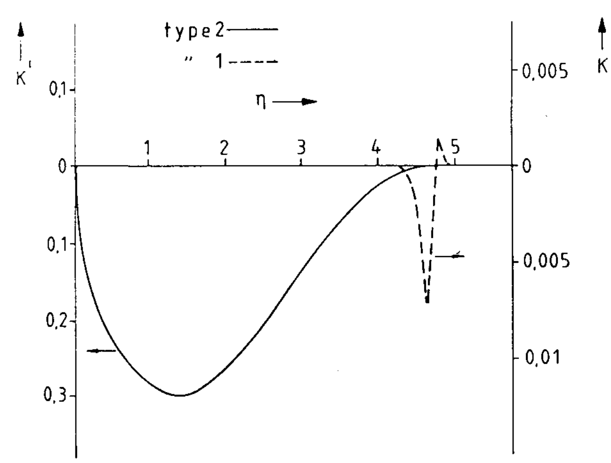

Fig. 9 Unstable eigenvectors of the case 1 kinetic energy profile for the round jet.

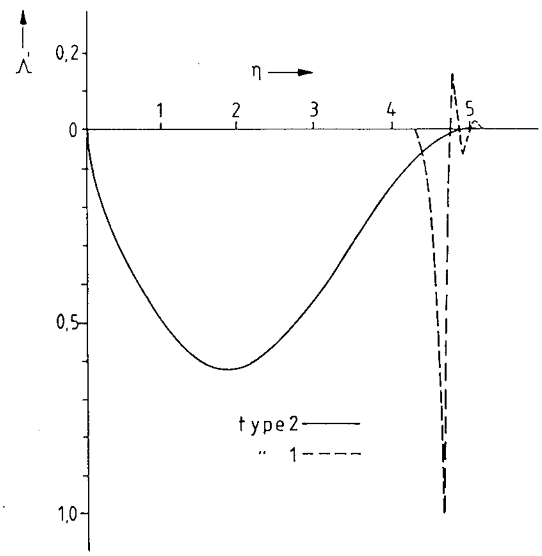

Fig. 10 Unstable eigenvectors of the case 1 turbulent viscosity profile for the round jet.

where μ is complex with a real part

$$\text{Real}(\mu) = \tfrac{1}{2}(3n+1) + E_2 m(\alpha - c_{\epsilon 2}) \quad (37)$$

Typical values are $E_2 = 1.8$, $m = 1.2$, $\alpha = 1.0$, and $c_{\epsilon 2} = 1.9$, leading to an approximately vanishing real part of μ and, therefore, to marginal instability for the round jet with $n = 1$, whereas the solution of the plane jet with $n = \tfrac{1}{2}$ is stable. Small changes in the parameters would considerably influence the stability of the solution.

The Stability Analysis

A large number of numerical calculations was performed with different combinations of the parameters used. The linear stability analysis consisted in perturbing all similar profiles including F and U in the form of Eq. (30) and discretising the resulting perturbation equation by using finite-difference formulas to obtain a matrix eigenvalue problem of the form

$$Ax = \mu Bx$$

when x is the vector of $F_i, U_i, \kappa_i, \Lambda_i$ $(1 = 2,\ldots,n-1)$. Here, $n = 42$ was the number of mesh points of the discretization.

The eigenvalue problem was solved using a standard package and the resulting eigenvalues were analyzed. All results showed at least one positive eigenvalue and, therefore, were unstable. The eigenvectors corresponding to the positive eigenvalues were also analyzed. It was found that there existed two typical shapes of the eigenvectors. As an example, in the k-ϵ model calculation case 1 with $c_\lambda = 0.025$, two real positive eigenvalues (out of 160), $\mu_1 = 1782.5$ and $\mu_2 = 1.13$ were found for the round jet and one real positive eigenvalue for the plane jet, $\mu_1 = 420.0$. The corresponding eigenvectors, e.g., for the round jet, are shown in Figs. 8-10. The first type eigenvectors belonging to μ_1 are very small for $\eta < 4$ but blow up at around $\eta = 4.5$ with an oscillatory behavior for larger η. The largest amplification of the instability clearly occurs in the region where the profiles decay to zero. The second type of eigenvectors corresponding to the second eigenvalue have a maximum for $\eta < 2.0$ and decay with small oscillations to large η. Only the second type was observed in all round jet calculations, but not in the plane jet. It was found that the eigenvalues corresponding to the first type increased strongly with decreasing c_λ, while the second type eigenvalues were independent of c_λ. For instance, in a k-ϵ model calculation with $c_\lambda = 0.05$, the type 1 instability had vanished and only one positive eigenvalue was found with an eigenvector of type 2. The same situation is found in the Rotta model calculation (case 2) with $c_\lambda = 0.05$.

An inspection of the steady profiles shows a more rapid change in the derivatives around $\eta = 4.5$ in case 1 than in the other case. Therefore, we believe that the type 1 instability originates from the wave-like character of the solution. This is supported by the fact that it disappears with increasing c_λ. The other remaining instability for the round jet seems to correspond to the global instability analyzed in the preceding paragraph. It originates essentially from the different power law of the centerline decay of k and ϵ ($k_c \simeq \xi^{-2}$, $\epsilon_c \simeq \xi^{-4}$), which leads to the first term in Eq. (37).

Conclusion

Two types of instabilities have been identified in free turbulent shear flow calculations. The first type results from the wave-like character of the equations and develops in the region where the turbulence decays discontinuously to zero at the outer edge. This instability appears in the round jet as well as in the plane jet. The second type, only observed in the round jet, appears to be inherent to the k-ϵ model and may be attributed to the different power law decay of these two quantities. Both instabilities appear to be weak. This also explains the success of numerical damping and of special manipulations in the iteration procedure. There is no doubt, however, that a set of modeled equations that do not provide a stable solution is unsatisfactory. It seems necessary to develop special numerical techniques to take the degenerate character of the equations into account.

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